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Complexified Gravity in Noncommutative Spaces

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ABSTRACT

The presence of a constant background antisymmetric tensor for open strings or D-branes forces the space-time coordinates to be noncommutative. This effect is equivalent to replacing ordinary products in the effective theory by the deformed star product. An immediate consequence of this is that all fields get complexified. The only possible noncommutative Yang-Mills theory is the one with $U(N)$ gauge symmetry. By applying this idea to gravity one discovers that the metric becomes complex. We show in this article that this procedure is completely consistent and one can obtain complexified gravity by gauging the symmetry $U(1, D-1)$ instead of the usual $SO(1, D-1)$. The final theory depends on a Hermitian tensor containing both the symmetric metric and antisymmetric tensor. In contrast to other theories of nonsymmetric gravity the action is both unique and gauge invariant. The results are then generalized to noncommutative spaces.

1 Introduction

The developments in the last two years have shown that the presence of a constant background B-field for open strings or D-branes lead to the noncommutativity of space-time coordinates ([1],[2],[3], [4],[5],[6],[7]). This can be equivalently realized by deforming the algebra of functions on the classical world volume. The operator product expansion for vertex operators is identified with the star (Moyal) product of functions on noncommutative spaces ([8],[9]). In this respect it was shown that noncommutative U(N) Yang-Mills theory does arise in string theory.

The effective action in presence of a constant B-field background is

$$-\frac{1}{4}\int Tr\left(F_{\mu\nu} * F^{\mu\nu}\right)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + iA_\mu * A_\nu - iA_\nu * A_\mu$$

and the star product is defined by

$$f(x) * g(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\zeta^\mu}\frac{\partial}{\partial\eta^\nu}} f(x+\zeta)\, g(x+\eta)\,|_{\zeta=\eta=0}$$

This definition forces the gauge fields to become complex. Indeed the noncommutative Yang-Mills action is invariant under the gauge transformations

$$A_\mu^g = g * A_\mu * g_*^{-1} - \partial_\mu g * g_*^{-1}$$

where $g_*^{-1}$is the inverse of g with respect to the star product:

$$g * g_*^{-1} = g_*^{-1} * g = 1$$

The contributions of the terms $i\theta^{\mu\nu}$ in the star product forces the gauge fields to be complex. Only conditions such as $A_\mu^\dagger = -A_\mu$ could be preserved under gauge transformations provided that g is unitary: $g^\dagger * g = g * g^\dagger = 1$. It is not possible to restrict A_μ to be real or imaginary to get the orthogonal or symplectic gauge groups as these properties are not preserved by the star product ([7],[10]). I will address the question of how is gravity modified in the low-energy effective theory of open strings in the presence of background fields. It has been shown that the metric of the target space gets modified by contributions of the B-field and that it becomes nonsymmetric ([11],[7]). If we think of gravity as resulting from local

gauge invariance under Lorentz transformations in the tangent manifold, then the previous reasoning would suggest that the vielbein and spin connection both get complexified with the star product. This seems inevitable as the star product appears in the operator product expansion of the string vertex operators.

We are therefore led to investigate whether gravity in D dimensions can be constructed by gauging the unitary group $U(1, D-1)$. In this article we shall show that this is indeed possible and that one can construct a Hermitian action which governs the dynamics of a nonsymmetric complex metric. Once this is achieved, it is straightforward to give the necessary modifications to make the action noncommutative. The plan of this paper is as follows. In section two the action for nonsymmetric gravity based on gauging the group $U(1, D-1)$ is given and the structure of the theory studied. In section three the equations of motion are solved to make connection with the second order formalism. In section four we give the generalization to noncommutative spaces. Section five is the conclusion.

2 Nonsymmetric gravity by gauging U(1,D-1)

Assume that we start with the $U(1, D-1)$ gauge fields $\omega_{\mu\ b}^{\ a}$. The $U(1, D-1)$ group of transformations is defined as the set of matrix transformations leaving the quadratic form

$$\left(Z^a\right)^\dagger \eta_b^a Z^b$$

invariant, where Z^a are D complex fields and

$$\eta_b^a = diag\left(-1, 1, \cdots, 1\right)$$

with $D-1$ positive entries. The gauge fields $\omega_{\mu\ b}^{\ a}$ must then satisfy the condition

$$\left(\omega_{\mu\ b}^{\ a}\right)^\dagger = -\eta_c^b \omega_{\mu\ d}^{\ c} \eta_a^d$$

The curvature associated with this gauge field is

$$R_{\mu\nu\ b}^{\ \ a} = \partial_\mu \omega_{\nu\ b}^{\ a} - \partial_\nu \omega_{\mu\ b}^{\ a} + \omega_{\mu\ c}^{\ a} \omega_{\nu\ b}^{\ c} - \omega_{\nu\ c}^{\ a} \omega_{\mu\ b}^{\ c}$$

Under gauge transformations we have

$$\widetilde{\omega}_{\mu\ b}^{\ a} = M_c^a \omega_{\mu\ d}^{\ c} M_b^{-1d} - M_c^a \partial_\mu M_b^{-1c}$$

where the matrices M are subject to the condition:

$$\left(M_c^a\right)^\dagger \eta_b^a M_d^b = \eta_d^c$$

The curvature then transforms as

$$\widetilde{R}_{\mu\nu}{}^a{}_b = M_c^a R_{\mu\nu}{}^c{}_d M_b^{-1d}$$

Next we introduce the complex vielbein e_μ^a and its inverse e_a^μ defined by

$$\begin{aligned} e_a^\nu e_\mu^a &= \delta_\mu^\nu \\ e_\nu^a e_b^\nu &= \delta_b^a \end{aligned}$$

which transform as

$$\begin{aligned} \widetilde{e}_\mu^a &= M_b^a e_\mu^b \\ \widetilde{e}_a^\mu &= \widetilde{e}_b^\mu M_a^{-1b} \end{aligned}$$

It is also useful to define the complex conjugates

$$\begin{aligned} e_{\mu a} &\equiv \left(e_\mu^a\right)^\dagger \\ e^{\mu a} &\equiv \left(e_a^\mu\right)^\dagger \end{aligned}$$

With this, it is not difficult to see that

$$e_a^\mu R_{\mu\nu}{}^a{}_b \eta_c^b e^{\nu c}$$

transforms to

$$e_d^\mu M_a^{-1d} M_e^a R_{\mu\nu}{}^e{}_f M_b^{-1f} \eta_c^b \left(M_c^{-1l}\right)^\dagger e^{\nu l}$$

and is thus $U(1, D-1)$ invariant. It is also Hermitian

$$\left(e_a^\mu R_{\mu\nu}{}^a{}_b \eta_c^b e^{\nu c}\right)^\dagger = -e_c^\nu \eta_b^c \eta_e^b R_{\mu\nu}{}^e{}_f \eta_a^f e^{\mu a} = e_a^\mu R_{\mu\nu}{}^a{}_b \eta_c^b e^{\nu c}$$

The metric is defined by

$$g_{\mu\nu} = \left(e_\mu^a\right)^\dagger \eta_b^a e_\nu^b$$

satisfy the property

$$g_{\mu\nu}^\dagger = g_{\nu\mu}$$

When the metric is decomposed into its real and imaginary parts:

$$g_{\mu\nu} = G_{\mu\nu} + iB_{\mu\nu}$$

the hermiticity property then implies the symmetries

$$\begin{aligned} G_{\mu\nu} &= G_{\nu\mu} \\ B_{\mu\nu} &= -B_{\nu\mu} \end{aligned}$$

The gauge invariant Hermitian action is given by

$$I = \int d^D x \sqrt{G} e_a^\mu R_{\mu\nu}{}^a{}_b \eta_c^b e^{\nu c}$$

This action is analogous to the first order formulation of gravity obtained by gauging the group $SO(1, D-1)$ One goes to the second order formalism by integrating out the spin connection and substituting for it its value in terms of the vielbein. The same structure is also present here and one can solve for $\omega_\mu{}^a{}_b$ in terms of the complex fields e_μ^a resulting in an action that depends only on the fields $g_{\mu\nu}$. It is worthwhile to stress that the above action, unlike others proposed to describe nonsymmetric gravity [12] is unique, except for the measure, and unambiguous. Similar ideas have been proposed in the past based on gauging the groups $O(D, D)$ [13] and $GL(D)$ [14], in relation to string duality, but the results obtained there are different from what is presented here. The ordering of the terms in writing the action is done in a way that generalizes to the noncommutative case.

The infinitesimal gauge transformations for e_μ^a is

$$\delta e_\mu^a = \Lambda_b^a e_\mu^b$$

which can be decomposed into real and imaginary parts by writing $e_\mu^a = e_{0\mu}^a + ie_{1\mu}^a$, and $\Lambda_b^a = \Lambda_{0b}^a + i\Lambda_{1b}^a$ to give

$$\begin{aligned} \delta e_{0\mu}^a &= \Lambda_{0b}^a e_{0\mu}^b - \Lambda_{1b}^a e_{1\mu}^b \\ \delta e_{1\mu}^a &= \Lambda_{1b}^a e_{0\mu}^b + \Lambda_{0b}^a e_{1\mu}^b \end{aligned}$$

The gauge parameters satisfy the constraints $(\Lambda_b^a)^\dagger = -\eta_c^b \Lambda_d^c \eta_a^d$ which implies the two constraints

$$\begin{aligned} (\Lambda_{0b}^a)^T &= -\eta_c^b \Lambda_{0d}^c \eta_a^d \\ (\Lambda_{1b}^a)^T &= \eta_c^b \Lambda_{1d}^c \eta_a^d \end{aligned}$$

From the gauge transformations of $e_{0\mu}^a$ and $e_{1\mu}^a$ one can easily show that the gauge parameters Λ_{0b}^a and Λ_{1b}^a can be chosen to make $e_{0\mu a}$ symmetric in μ and

a and $e_{1\mu a}$ antisymmetric in μ and a. This is equivalent to the statement that the Lagrangian should be completely expressible in terms of $G_{\mu\nu}$ and $B_{\mu\nu}$ only, after eliminating $\omega_{\mu\ b}^{\ a}$ through its equations of motion. In reality we have

$$\begin{aligned} G_{\mu\nu} &= e_{0\mu}^a e_{0\nu}^b \eta_{ab} + e_{1\mu}^a e_{1\nu}^b \eta_{ab} \\ B_{\mu\nu} &= e_{0\mu}^a e_{1\nu}^b \eta_{ab} - e_{1\mu}^a e_{0\nu}^b \eta_{ab} \end{aligned}$$

In this special gauge, where we define $g_{0\mu\nu} = e_{0\mu}^a e_{0\nu}^b \eta_{ab}$, $g_{0\mu\nu} g_0^{\nu\lambda} = \delta_\mu^\lambda$, and use $e_{0\mu}^a$ to raise and lower indices we get

$$\begin{aligned} B_{\mu\nu} &= -2e_{1\mu\nu} \\ G_{\mu\nu} &= g_{0\mu\nu} - \frac{1}{4} B_{\mu\kappa} B_{\lambda\nu} g_0^{\kappa\lambda} \end{aligned}$$

The last formula appears in the metric of the effective action in open string theory [11].

3 Second Order Formulation

We can express the Lagrangian in terms of e_μ^a only by solving the $\omega_{\mu\ b}^{\ a}$ equations of motion

$$\begin{aligned} e_a^\mu e^{\nu b} \omega_{\nu\ b}^{\ c} + e_b^\nu e^{\mu c} \omega_{\nu\ a}^{\ b} - e^{\mu b} e_a^\nu \omega_{\nu\ b}^{\ c} - e_b^\mu e^{\nu c} \omega_{\nu\ a}^{\ b} &= \\ \frac{1}{\sqrt{G}} \partial_\nu \left(\sqrt{G} \left(e_a^\nu e^{\mu c} - e_a^\mu e^{\nu c} \right) \right) &\equiv X^{\mu c}{}_a \end{aligned}$$

where $X^{\mu c}{}_a$ satisfy $(X^{\mu c}{}_a)^\dagger = -X^{\mu a}{}_c$. One has to be very careful in working with a nonsymmetric metric

$$\begin{aligned} g_{\mu\nu} &= e_\mu^a e_{\nu a} \\ g^{\mu\nu} &= e^{\mu a} e_{\nu a} \\ g_{\mu\nu} g^{\nu\rho} &= \delta_\mu^\rho \end{aligned}$$

but $g_{\mu\nu} g^{\mu\rho} \neq \delta_\mu^\rho$. Care also should be taken when raising and lowering indices with the metric.

Before solving the ω equations, we point out that the trace part of $\omega_{\mu\ b}^{\ a}$ (corresponding to the $U(1)$ part in $U(D)$) must decouple from the other gauge fields. It is thus undetermined and decouples from the Lagrangian after substituting its equation of motion. It imposes a condition on the e_μ^a

$$\frac{1}{\sqrt{G}} \partial_\nu \left(\sqrt{G} \left(e_a^\nu e^{\mu a} - e_a^\mu e^{\nu a} \right) \right) \equiv X^{\mu a}{}_a = 0$$

We can therefore assume, without any loss in generality, that $\omega_{\mu\ b}^{\ a}$ is traceless $\left(\omega_{\mu\ a}^{\ a}=0\right)$.

Multiplying the $\omega-$equation with $e_a^{\kappa}e_c^{\rho}$ we get

$$\delta_{\kappa}^{\mu}\omega_{\nu\rho}^{\quad\nu}+\delta_{\rho}^{\mu}\omega_{\nu\ \kappa}^{\ \nu}-\omega_{\kappa\rho}^{\quad\mu}-\omega_{\rho\ \kappa}^{\ \mu}=X^{\mu}_{\ \rho\kappa}$$

where

$$\begin{aligned}\omega_{\mu\nu}^{\quad\rho} &= e_{\nu a}e^{\rho b}\omega_{\mu\ b}^{\ a}\\ X^{\mu}_{\ \rho\kappa} &= e_{\rho c}e_{\kappa}^{a}X^{\mu c}_{\quad a}\end{aligned}$$

Contracting by first setting $\mu=\kappa$ then $\mu=\rho$ we get the two equations

$$\begin{aligned}3\omega_{\nu\rho}^{\quad\nu}+\omega_{\nu\ \rho}^{\ \nu} &= X^{\mu}_{\ \rho\mu}\\ \omega_{\nu\rho}^{\quad\nu}+3\omega_{\nu\ \rho}^{\ \nu} &= X^{\mu}_{\ \mu\rho}\end{aligned}$$

These could be solved to give

$$\begin{aligned}\omega_{\nu\rho}^{\quad\nu} &= \frac{1}{8}\left(3X^{\mu}_{\ \rho\mu}-X^{\mu}_{\ \mu\rho}\right)\\ \omega_{\nu\ \rho}^{\ \nu} &= \frac{1}{8}\left(-X^{\mu}_{\ \rho\mu}+3X^{\mu}_{\ \mu\rho}\right)\end{aligned}$$

Substituting these back into the $\omega-$equation we get

$$\omega_{\kappa\rho}^{\quad\mu}+\omega_{\rho\ \kappa}^{\ \mu}=\frac{1}{8}\delta_{\kappa}^{\mu}\left(3X^{\mu}_{\ \rho\mu}-X^{\mu}_{\ \mu\rho}\right)+\frac{1}{8}\delta_{\rho}^{\mu}\left(-X^{\mu}_{\ \kappa\mu}+3X^{\mu}_{\ \mu\kappa}\right)-X^{\mu}_{\ \rho\kappa}\equiv Y^{\mu}_{\ \rho\kappa}$$

We can rewrite this equation after contracting with $e_{\mu c}e_{\sigma}^{c}$ to get

$$\omega_{\kappa\rho\sigma}+e_a^{\mu}e_{\mu c}e_{\sigma}^{c}\omega_{\rho\ \kappa}^{\ a}=g_{\sigma\mu}Y^{\mu}_{\ \rho\kappa}\equiv Y_{\sigma\rho\kappa}$$

By writing $\omega_{\rho\ \kappa}^{\ a}=\omega_{\rho\nu\kappa}e^{\nu a}$ we finally get

$$\left(\delta_{\kappa}^{\alpha}\delta_{\rho}^{\beta}\delta_{\sigma}^{\gamma}+g^{\beta\mu}g_{\sigma\mu}\delta_{\rho}^{\alpha}\delta_{\kappa}^{\gamma}\right)\omega_{\alpha\beta\gamma}=Y_{\sigma\rho\kappa}$$

To solve this equation we have to invert the tensor

$$M_{\kappa\rho\sigma}^{\alpha\beta\gamma}=\delta_{\kappa}^{\alpha}\delta_{\rho}^{\beta}\delta_{\sigma}^{\gamma}+g^{\beta\mu}g_{\sigma\mu}\delta_{\rho}^{\alpha}\delta_{\kappa}^{\gamma}$$

In the conventional case when all fields are real, the metric $g_{\mu\nu}$ is symmetric and $g^{\beta\mu}g_{\sigma\mu}=\delta_{\sigma}^{\beta}$ so that the inverse of $M_{\kappa\rho\sigma}^{\alpha\beta\gamma}$ is simple. In the present case, because of the nonsymmetry of $g_{\mu\nu}$ this is fairly complicated and could only be

solved by a perturbative expansion. Writing $g_{\mu\nu} = G_{\mu\nu} + iB_{\mu\nu}$ and from the definition $g^{\mu\nu} g_{\nu\rho} = \delta^\rho_\mu$ we get

$$g^{\mu\nu} = a^{\mu\nu} + ib^{\mu\nu}$$

where

$$\begin{aligned} a^{\mu\nu} &= \left(G_{\mu\nu} + B_{\mu\kappa} G^{\kappa\lambda} B_{\lambda\nu}\right)^{-1} \\ &= G^{\mu\nu} - G^{\mu\kappa} B_{\kappa\lambda} G^{\lambda\sigma} B_{\sigma\eta} G^{\eta\nu} + O(B^4) \end{aligned}$$

$$b^{\mu\nu} = -G^{\mu\kappa} B_{\kappa\lambda} G^{\lambda\nu} + G^{\mu\kappa} B_{\kappa\lambda} G^{\lambda\sigma} B_{\sigma\tau} G^{\tau\rho} B_{\rho\eta} G^{\eta\nu} + O(B^5)$$

We have defined $G^{\mu\nu} G_{\nu\rho} = \delta^\mu_\rho$. This implies that

$$\begin{aligned} g^{\mu\alpha} g_{\nu\alpha} &\equiv \delta^\mu_\nu + L^\mu_\nu \\ L^\mu_\nu &= iG^{\mu\rho} B_{\rho\nu} - 2G^{\mu\rho} B_{\rho\sigma} G^{\sigma\alpha} B_{\alpha\nu} + O(B^3) \end{aligned}$$

The inverse of $M^{\alpha\beta\gamma}_{\kappa\rho\sigma}$ defined by

$$N^{\sigma\rho\kappa}_{\alpha\beta\gamma} M^{\alpha'\beta'\gamma'}_{\sigma\rho\kappa} = \delta^{\alpha'}_\alpha \delta^{\beta'}_\beta \delta^{\gamma'}_\gamma$$

is evaluated to give

$$\begin{aligned} N^{\sigma\rho\kappa}_{\alpha\beta\gamma} &= \frac{1}{2}\left(\delta^\sigma_\gamma \delta^\rho_\beta \delta^\kappa_\alpha + \delta^\sigma_\beta \delta^\rho_\alpha \delta^\kappa_\gamma - \delta^\sigma_\alpha \delta^\rho_\gamma \delta^\kappa_\beta\right) \\ &\quad -\frac{1}{4}\left(\delta^\kappa_\beta \delta^\sigma_\alpha L^\rho_\gamma + \delta^\kappa_\alpha \delta^\sigma_\gamma L^\rho_\beta - \delta^\kappa_\gamma \delta^\sigma_\beta L^\rho_\alpha\right) \\ &\quad +\frac{1}{4}\left(L^\kappa_\gamma \delta^\sigma_\beta \delta^\rho_\alpha + L^\kappa_\beta \delta^\sigma_\alpha \delta^\rho_\gamma - L^\kappa_\alpha \delta^\sigma_\gamma \delta^\rho_\beta\right) \\ &\quad -\frac{1}{4}\left(\delta^\kappa_\alpha L^\sigma_\gamma \delta^\rho_\beta + \delta^\kappa_\gamma L^\sigma_\beta \delta^\rho_\alpha - \delta^\kappa_\beta L^\sigma_\alpha \delta^\rho_\gamma\right) + O(L^2) \end{aligned}$$

This enables us to write

$$\omega_{\alpha\beta\gamma} = N^{\sigma\rho\kappa}_{\alpha\beta\gamma} Y_{\rho\sigma\kappa}$$

and finally

$$\omega_{\mu\ b}^{\ a} = e^{\beta a} e^\gamma_b \omega_{\mu\beta\gamma}$$

It is clear that the leading term reproduces the Einstein-Hilbert action plus contributions proportional to $B_{\mu\nu}$ and higher order terms. The most difficult task is to show that the Lagrangian is completely expressible in terms of $G_{\mu\nu}$ and $B_{\mu\nu}$ only. The other components of $e^a_{0\mu}$ and $e^a_{1\mu}$ should disappear. We have argued from the view point of gauge invariance that this must happen, but it

will be nice to verify this explicitly, to leading orders. We can check that in the flat approximation for gravity with $G_{\mu\nu}$ taken to be $\delta_{\mu\nu}$, the $B_{\mu\nu}$ field gets the correct kinetic terms. First we write

$$
\begin{aligned}
e_\mu^a &= \delta_\mu^a - \frac{i}{2} B_{\mu a} \\
e_{\mu a} &= \delta_\mu^a + \frac{i}{2} B_{\mu a}
\end{aligned}
$$

and the inverses

$$
\begin{aligned}
e^{\mu a} &= \delta_\mu^a - \frac{i}{2} B_{\mu a} \\
e_a^\mu &= \delta_\mu^a + \frac{i}{2} B_{\mu a}
\end{aligned}
$$

The ${\omega_\mu}^a{}_a$equation implies the constraint

$$
X^{\mu a}{}_a = \partial_\nu \left(e_a^\mu e^{\nu a} - e_a^\nu e^{\mu a} \right) = 0
$$

This gives the gauge fixing condition

$$
\partial^\nu B_{\mu\nu} = 0
$$

We then evaluate

$$
X^\mu_{\rho\kappa} = -\frac{i}{2} \left(\partial_\rho B_{\kappa\mu} + \partial_\kappa B_{\rho\mu} \right)
$$

This together with the gauge condition on $B_{\mu\nu}$ gives

$$
Y^\mu_{\rho\kappa} = \frac{i}{2} \left(\partial_\rho B_{\kappa\mu} + \partial_\kappa B_{\rho\mu} \right)
$$

and finally

$$
\omega_{\mu\nu\rho} = -\frac{i}{2} \left(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\mu\rho} \right)
$$

When the $\omega_{\mu\nu\rho}$ is substituted back into the Lagrangian, and after integration by parts one gets

$$
\begin{aligned}
L &= \omega_{\mu\nu\rho} \omega^{\nu\rho\mu} - {\omega_\mu}^{\mu\rho} {\omega_{\nu\rho}}^\nu \\
&= -\frac{1}{4} B_{\mu\nu} \partial^2 B^{\mu\nu}
\end{aligned}
$$

This is identical to the usual expression

$$
\frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}
$$

where

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$$

We have therefore shown that in D dimensions one must start with $2D^2$ real components e^a_μ, subject to gauge transformations with D^2 real parameters. The resulting Lagrangian depends on D^2 fields, with $\frac{D(D+1)}{2}$ symmetric components $G_{\mu\nu}$ and $\frac{D(D-1)}{2}$ antisymmetric components $B_{\mu\nu}$.

4 Noncommutative Gravity

At this stage, and having shown that it is perfectly legitimate to formulate a theory of gravity with nonsymmetric complex metric, based on the idea of gauge invariance of the group $U(1, D-1)$. It is not difficult to generalize the steps that led us to the action for complex gravity to spaces where coordinates do not commute, or equivalently, where the usual products are replaced with star products.

First the gauge fields are subject to the gauge transformations

$$\widetilde{\omega}_{\mu\ b}^{\ a} = M^a_c * \omega_{\mu\ d}^{\ c} * M^{-1d}_{*b} - M^a_c * \partial_\mu M^{-1c}_{*b}$$

where M^{-1b}_{*a} is the inverse of M^a_b with respect to the star product. The curvature is now

$$R_{\mu\nu\ b}^{\ \ a} = \partial_\mu \omega_{\nu\ b}^{\ a} - \partial_\nu \omega_{\mu\ b}^{\ a} + \omega_{\mu\ c}^{\ a} * \omega_{\nu\ b}^{\ c} - \omega_{\nu\ c}^{\ a} * \omega_{\mu\ b}^{\ c}$$

which transforms according to

$$\widetilde{R}_{\mu\nu\ b}^{\ \ a} = M^a_c * R_{\mu\nu\ d}^{\ \ c} * M^{-1d}_{*b}$$

Next we introduce the vielbeins e^a_μ and their inverse defined by

$$\begin{aligned} e^\nu_{*a} * e^a_\mu &= \delta^\nu_\mu \\ e^a_\nu * e^\nu_{*b} &= \delta^a_b \end{aligned}$$

which transform to

$$\begin{aligned} \widetilde{e}^a_\mu &= M^a_b * e^b_\mu \\ \widetilde{e}^\mu_{*a} &= \widetilde{e}^\mu_b * M^{-1b}_{*a} \end{aligned}$$

The complex conjugates for the vielbeins are defined by

$$\begin{aligned} e_{\mu a} &\equiv \left(e_{\mu}^{a}\right)^{\dagger} \\ e_{*}^{\mu a} &\equiv \left(e_{*a}^{\mu}\right)^{\dagger} \end{aligned}$$

Finally we define the metric

$$g_{\mu\nu} = \left(e_{\mu}^{a}\right)^{\dagger} \eta_{b}^{a} * e_{\nu}^{b}$$

The $U(1, D-1)$ gauge invariant Hermitian action is

$$I = \int d^{D}x\sqrt{G}\left(e_{*a}^{\mu} * R_{\mu\nu}{}^{a}{}_{b}\eta_{c}^{b} * e_{*}^{\nu c}\right)$$

This action differs from the one considered in the commutative case by higher derivatives terms proportional to $\theta^{\mu\nu}$. It would be very interesting to see whether these terms could be reabsorbed by redefining the field $B_{\mu\nu}$, or whether the Lagrangian reduces to a function of $G_{\mu\nu}$ and $B_{\mu\nu}$ and their derivatives only.

The connection of this action to the gravity action derived for noncommutative spaces based on spectral triples ([15],[16],[17]) remains to be made. In order to do this one must understand the structure of Dirac operators for spaces with deformed star products.

5 Conclusions

We have shown that it is possible to combine the tensors $G_{\mu\nu}$ and $B_{\mu\nu}$ into a complexified theory of gravity in D dimensions by gauging the group $U(1, D-1)$. The Hermitian gauge invariant action is a direct generalization of the first order formulation of gravity obtained by gauging the Lorentz group $SO(1, D-1)$. The Lagrangian obtained is a function of the complex fields e_{μ}^{a} and reduces to a function of $G_{\mu\nu}$ and $B_{\mu\nu}$ only. This action is generalizable to noncommutative spaces where coordinates do not commute, or equivalently, where the usual products are deformed to star products. It is remarkable that the presence of a constant background field in open string theory implies that the metric of the target space becomes nonsymmetric and that the tangent manifold for space-time does not have only the Lorentz symmetry but the larger $U(1, D-1)$ symmetry. The results shown here, can be improved by computing the second order action to include higher order terms in the $B_{\mu\nu}$ expansion and to see if

this can be put in a compact form. Similarly the computation has to be repeated in the noncommutative case to see whether the $\theta^{\mu\nu}$ contributions could be simplified. It is also important to determine a link between this formulation of noncommutative gravity and the Connes formulation based on the noncommutative geometry of spectral triples. To make such connection many points have to be clarified, especially the structure of the Dirac operator for such a space. This and other points will be explored in future publication

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